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Divisibility Properties of Group Rings over Torsion-free Abelian Groups

by

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Abstract. Let G be a torsion-free abelian group of type $(0, 0, 0, \cdots)$ and R an integrally closed integral domain with quotient field K. We show that every divisorial ideal (respectively, t-ideal) J of the group ring $R[X; G]$ is of the form $J = hIR[X; G]$ for some $h \in K[X; G]$ and a divisorial ideal (respectively, t-ideal) I of R. Consequently, there are natural monoid isomorphisms $Cl(R) \cong Cl(R[X; G])$ and $Cl_t(R) \cong Cl_t(R[X; G])$.

Throughout this paper, G shall stand for a torsion-free abelian group, R an integral domain, and K quotient field of R. Gilmer and Parker [5, Theorem 7.13] proved that $R[X; G]$ is a unique factorization domain (UFD) if and only if R is a UFD and G is of type $(0, 0, 0, \cdots)$. Using this result, Matsuda [8, Proposition 3.3] proved that $R[X; G]$ is a Krull domain if and only if R is a Krull domain and G is of type $(0, 0, 0, \cdots)$. In [9, Proposition 1 and 6 §12], Matsuda proved that if $R[X; G]$ is a π-domain, then R is a π-domain and G is of type $(0, 0, 0, \cdots)$. Later, in [1, Proposition 6.5], D. D. Anderson and D. F. Anderson proved that $R[X; G]$ is a π-domain if and only if R is a π-domain and G is of type $(0, 0, 0, \cdots)$. In [2, Proposition 6.5], it was also shown that the semigroup ring $R[X; \Gamma]$ is a PVMD if and only if R is a PVMD and Γ is a PVMD semigroup. This is an answer to Malik's question [7]. In [7], Malik studied divisorial properties of the semigroup ring $R[X; \Gamma]$. One of her results is that if R is an integrally closed domain and J is a divisorial ideal of the semigroup ring $R[X; \Gamma]$ such that $J \cap R \neq (0)$, then J is extended from R [7, Theorem 5.1.7]. Another result [7, Proposition 5.1.4] is that if R is an integrally closed domain, then $(A_{fg})_v = (A_f A_g)_v$ for all $f, g \in R[X; G]$. Using Malik's results and Gilmer & Parker's result, we will show that if R is integrally closed and G is of type $(0, 0, 0, \cdots)$, then every divisorial ideal J of $R[X; G]$ is of the form $J = hIR[X; G]$ where $h \in K[X; G]$ and I is a divisorial ideal of R. Thus we have isomorphisms between monoids: $Cl(R[X; G]) \cong Cl(R)$ and $Cl_t(R[X; G]) \cong Cl_t(R)$ naturally. This is the generalization of the well-known fact for $G = \mathbb{N}$ which gives unified proofs of the facts that R is a GCD domain, a UFD, a Krull domain, a π-domain, a Prüfer v-multiplication domain if and only if $R[X]$ is. For R is a GCD domain $\Leftrightarrow$ every finite type t-ideal is principal, R is a UFD $\Leftrightarrow$ $Cl_t(R) = 0$, R is a Krull domain $\Leftrightarrow$ every nonzero ideal is t-invertible, and R is a π-domain $\Leftrightarrow$ every nonzero t-ideal is invertible. Similarly, using this generalization, we will give new proofs of several known results including the above mentioned ones as well as new results. This will explain why the group ring $R[X; G]$ behaves like the polynomial ring $R[X]$ in many cases related to divisorial properties. For undefined terms and notations, the readers are referred to [4].

For a fractional ideal I of R, I_v is defined as the fractional ideal $(R :_K (R :_K I))$, and I_t is defined as the fractional ideal $\sum\{(I_o)_v \mid I_o$ is a finitely generated fractional ideal of R contained in $I\}$. In case $I_v = I$ [resp. $I_t = I$], I is called a divisorial ideal [resp. t-ideal].

LEMMA 1 ([7, Theorem 5.1.7]). *Let T be an integrally closed domain. Let I be a fractional t-ideal of $R[X; G]$ such that $I \subseteq K[X; G]$ and $I \cap K \neq (0)$. Then $I = (I \cap K)R[X; G]$.*

Proof. Since I is an $R[X; G]$-module, clearly $I \supseteq (I \cap K)R[X; G]$. For the reverse containment, let $a \in (I \cap K) \setminus (0)$, $f \in I \setminus (0)$. Let g be an arbitrary nonzero element of $(a, f)^{-1}$. Clearly $g \in K[X; G]$. By [7, Proposition 5.1.4], $(A_f A_g)_v = (A_{fg})_v \subseteq R$. Hence $g \in A_g R[X; G] \subseteq A_f^{-1} R[X; G]$. So $(a, f)^{-1} \subseteq A_f^{-1} R[X; G]$. Taking inverses and applying [7, Proposition 5.1.1], we get $(A_f)_v R[X; G] \subseteq (a, f)_v \subseteq I$. So $A_f \subseteq I$ and we deduce that $I \subseteq (I \cap K)R[X; G]$.

LEMMA 2. *Let R be an integrally closed domain. For any nonzero elements f, g in $R[X; G]$, there exists an $h \in K[X; G]$ and an integral divisorial ideal I of R such that $fR[X; G] \cap gR[X; G] = hIR[X; G]$.*

Proof. Let $S = R \setminus \{0\}$ and $J = fR[X; G] \cap gR[X; G]$. Since $K[X; G]$ is a GCD domain by [5, Theorem 5.2], $J_S = fK[X; G] \cap gK[X; G] = hK[X; G]$ for some $h \in K[X; G]$. So, $h^{-1}J \subseteq K[X; G]$, and $h \in J_S$ implies that $h^{-1}J \cap K \neq (0)$. Thus $A = h^{-1}J$ is a fractional t-ideal of $R[X; G]$ and $A \cap K \neq (0)$. Applying Lemma 1, we deduce that $A = (A \cap K)R[X; G]$. Thus $h^{-1}J = IR[X; G]$, for some R-submodule I of K. Now $J = hIR[X; G]$. Since $hI \subseteq J \subseteq R[X; G]$, $A_h I \subseteq R$. So we may assume that $I \subseteq R$. Finally $hIR[X; G]$ is a divisorial ideal and therefore, so is I.

Recall that a torsion-free abelian group is of type $(0, 0, 0, \cdots)$ if it satisfies the ascending chain condition on cyclic subgroups.

LEMMA 3. *Suppose that G is of type $(0, 0, 0, \cdots)$ and that R is integrally closed. Then every t-ideal J of $R[X; G]$ is of the form $J = hIR[X; G]$ for some $h \in K[X; G]$ and a t-ideal I of R.*

Proof. Let $S = \{fK[X; G] \mid f \in J\}$. Since $K[X; G]$ is a unique factorization domain [5, Theorem 7.12], S has a maximal element with respect to inclusion, say (f_0). Let $0 \neq f_\alpha \in J$. By Lemma 2, $(f_\alpha) \cap (f_0) = gHR[X; G]$ for $g \in K[X; G]$ and an integral divisorial ideal H of R. Now $(f_\alpha, f_0)_v = \left(\left(\frac{1}{f_\alpha}\right) \cap \left(\frac{1}{f_0}\right)\right)^{-1} = f_\alpha f_0((f_\alpha) \cap (f_0))^{-1} =$

$f_\alpha f_0(gHR[X; G])^{-1} = \frac{f_\alpha f_0}{g} H^{-1}R[X; G]$. Put $\frac{f_\alpha f_0}{g} = h_\alpha$ and $H^{-1} = I_\alpha$. Then $h_\alpha \in K[X; G]$, I_α is a fractional t-ideal of R, and $(f_\alpha, f_0)_v = h_\alpha I_\alpha R[X; G]$. Since $h_\alpha I_\alpha \subseteq (f_\alpha, f_0)_v \subseteq J$, we may assume that $h_\alpha \in J$ and I_α is a fractional t-ideal of R. Now $f_0 \in h_\alpha I_\alpha R[X; G]$ and so $f_0 K[X; G] \subseteq h_\alpha K[X; G]$. Hence $f_0 K[X; G] = h_\alpha K[X; G]$ by the maximality of f_0 in S. So $h_\alpha = f_0 u X^\beta$ for $u \in K$, $\beta \in G$ by [5, Corollary 4.2]. So, $(f_\alpha, f_0)_v = f_0 u X^\beta I_\alpha R[X; G] = f_0 u I_\alpha R[X; G]$ since X^β is a unit of $R[X; G]$. Thus $(f_\alpha, f_0)_v = f_0 I_\alpha R[X; G]$ for some R-submodule I_α of K. Put $I = \sum_\alpha I_\alpha$. Now $J = \sum_{f_\alpha \in J} (f_\alpha, f_0)_v = \sum_\alpha f_0 I_\alpha R[X; G] = f_0(\sum_\alpha I_\alpha) R[X; G] = f_0 I R[X; G]$. Since J is a t-ideal of $R[X; G]$, I is a (fractional) t-ideal of R.

For a torsion-free abelian group G, it is known that R is completely integrally closed if and only if $R[X; G]$ is ([8, Proposition 3.3.]). The class monoid $Cl(R)$ which is the fractional divisorial ideals modulo principal fractional ideals is a group if R is completely integrally closed. The t-class monoid $Cl_t(R)$ is defined as the fractional t-ideals modulo principal fractional ideals of R. There is a natural monoid homomorphism ϕ from the monoid $Cl(R)$ (resp. $Cl_t(R)$) to the monoid $Cl(R[X; G])$ (resp. $Cl_t(R[X; G])$) given by $\phi(cl(I)) = cl(IR[X; G])$ (resp. $\phi(cl_t(I)) = cl_t(IR[X; G])$). In case when ϕ is an isomorphism, we say that $Cl(R)$ (resp. $Cl_t(R)$) is naturally isomorphic to $Cl(R[X; G])$ (resp. $Cl_t(R[X; G])$). Let R be an integrally closed domain. Although $R[X; G]$ is also an integrally closed domain for any torsion-free abelian group G [8, Proposition 4.4], our next result shows that it is not necessarily true that $Cl_t(R) \cong Cl_t(R[X; G])$ naturally for all G, which holds when $G = \mathbb{N}$.

THEOREM. *The following statements are equivalent for an integral domain R and a torsion-free abelian group G.*

(1) $Cl_t(R) \cong Cl_t(R[X; G])$ *naturally.*

(2) $fK[X; G] \cap R[X; G] = fA_f^{-1}R[X; G]$ *for every* $f \in R[X; G]$ *and G is of type* $(0, 0, 0, \cdots)$.

(3) $(A_f A_g)_v = (A_{fg})_v$ *for all* $f, g \in R[X; G]$ *and G is of type* $(0, 0, 0, \cdots)$.

(4) *R is integrally closed and G is of type* $(0, 0, 0, \cdots)$.

Proof. (1) $\Rightarrow$ (2). Let H be a t-ideal of $K[X; G]$. Then $J = H \cap R[X; G]$ is a t-ideal of $R[X; G]$ since $K[X; G] = R[X; G]_{R^*}$. Then assumption implies that $J = hIR[X; G]$; $h \in K[X; G]$, I a t-ideal of R. Then $H = JK[X; G] = hK[X; G]$. So every t-ideal of $K[X; G]$ is principal, whence $K[X; G]$ is a UFD. From [5, Corollary 7.14], we deduce that G is of type $(0, 0, 0, \cdots)$. To verify the equality, let $f \in R[X; G]$. Since $fK[X; G] \cap R[X; G]$ is a t-ideal of $R[X; G]$, it is of the form $hIR[X; G]$ for $h \in K[X; G]$ and I a t-ideal of R. Then $fK[X; G] = hK[X; G]$. In view of [5, Corollary 4.2], $h = uX^\alpha f$ for $0 \neq u \in K$ and $\alpha \in G$. So $uA_f = A_h$. Now from $hK[X; G] \cap R[X; G] = hIR[X; G]$, it easily follows that $I = A_h^{-1}$. Thus $hK[X; G] \cap R[X; G] = hA_h^{-1}R[X; G] = ufX^\alpha(uA_f)^{-1}R[X; G] = ufX^\alpha u^{-1}(A_f)^{-1}R[X; G] = fX^\alpha(A_f)^{-1}R[X; G] = f(A_f)^{-1}R[X; G]$ since X^α is also a unit of $R[X; G]$.

(2) $\Rightarrow$ (3). An imitation of the polynomial case proof [11] will do the job: For

$f, g \in R[X; G]$, $fg(A_{fg})^{-1}R[X; G] = fgK[X; G] \cap R[X; G] \subseteq fK[X; G] \cap R[X; G] = f(A_f)^{-1}R[X; G] \Rightarrow fg(A_{fg})^{-1}R[X; G] \subseteq f(A_f)^{-1}R[X; G] \Rightarrow g(A_{fg})^{-1}R[X; G] \subseteq (A_f)^{-1}R[X; G] \Rightarrow A_g(A_{fg})^{-1} \subseteq (A_f)^{-1} \Rightarrow (A_f)_v A_g \subseteq (A_{fg})_v \Rightarrow (A_f A_g)_v = (A_{fg})_v$.

(3) $\Rightarrow$ (4). Since G is torsion-free, $R[X; \mathbb{N}]$ can be imbedded in $R[X; G]$. Then $(A_f A_g)_v = (A_{fg})_v$ for all $f, g \in R[X; \mathbb{N}]$. Now the problem is reduced to the polynomial case, and so R is integrally closed [11].

(4) $\Rightarrow$ (1). This is Lemma 3.

According to Matsuda [8, Proposition 4.3], $R[X; G]$ is completely integrally closed if and only if R is. In [8, Proposition 5.3], Matsuda showed that $Cl(R) \cong Cl(R[X; G])$ naturally provided that R is a Krull domain and G is of type $(0, 0, 0, \cdots)$. Our next result is a generalization of this. Note that Corollary 1 is the best we could do since for $Cl(R[X; G])$ to be a group, $R[X; G]$ has to be completely integrally closed.

COROLLARY 1. *If R is completely integrally closed and G is of type* $(0, 0, 0, \cdots)$, *then $Cl(R) \cong Cl(R[X; G])$ naturally as groups.*

Now Theorem can be used to give unified proofs of the following known results about divisorial relation between R and $R[X; G]$. An integral domain is called a Mori domain if it satisfies the ascending chain condition on divisorial ideals. For $G = \mathbb{N}$, it is well-known for an integrally closed domain R that R is a Mori domain $\Leftrightarrow R[X]$ is [11, Theorem 5]. We show that this also holds for any torsion-free abelian group G of type $(0, 0, 0, \cdots)$.

COROLLARY 2. *Let R be integrally closed. Then R is a Mori domain and G has type* $(0, 0, 0, \cdots)$ *if and only if $R[X; G]$ is a Mori domain.*

Proof. ($\Rightarrow$) Let J be a t-ideal of $R[X; G]$. By Lemma 3, $J = hIR[X; G]$ for $h \in K[X; G]$, a t-ideal I of R. Since R is a Mori domain, I is of finite type and so J is of finite type. Hence every t-ideal of $R[X; G]$ is of finite type and therefore $R[X; G]$ is a Mori domain.

($\Leftarrow$) Suppose that $R[X; G]$ is a Mori domain. Clearly a Mori domain satisfies the ascending chain condition on principal ideals. So G is of type $(0, 0, 0, \cdots)$ by [5, Corollary 7.14.].

COROLLARY 3 ([8, Proposition 3.3.]). *R is a Krull domain and G has the type* $(0, 0, 0, \cdots)$ *if and only if $R[X; G]$ is a Krull domain.*

Proof. ($\Rightarrow$) Let J be a t-ideal of $R[X; G]$. By Lemma 3, $J = hIR[X; G]$ for $h \in K[X; G]$, an integral t-ideal I of R. So $(JJ^{-1})_t = (II^{-1})_t R[X; G] = R[X; G]$. So $R[X; G]$ is a Krull domain [6, Theorem 3.6].

($\Leftarrow$) Suppose that $R[X; G]$ is a Krull domain. Then $R[X; G]$ satisfies the ascending chain condition on principal ideals. So G has the type $(0, 0, 0, \cdots)$ by [5, Corollary 7.14.]. From $R = R[X; G] \cap K$, we deduce that R is a Krull domain.

An integral domain R is called a π-domain if every principal ideal is a product of prime ideals [4]. R is a π-domain if and only if every nonzero t-ideal is invertible [6, Theorem 4.4].

COROLLARY 4 ([1, Proposition 6.5]). *R is a π-domain and G is of type* $(0, 0, 0, \cdots)$ *(if and) only if $R[X; G]$ is a π-domain.*

Proof. ($\Rightarrow$) Let J be a t-ideal of $R[X; G]$. By Lemma 3, $J = hIR[X; G]$ for an $h \in K[X; G]$, an integral t-ideal I of R. Now

$$JJ^{-1} = II^{-1}R[X; G] = R[X; G].$$

So $R[X; G]$ is a π-domain by [6, Theorem 4.4]. ($\Leftarrow$) Clearly, $R[X; G]$ satisfies the ascending chain condition on principal ideals. So G is of type $(0, 0, 0, \cdots)$. For a t-ideal $I \subseteq R$,

$$(II^{-1})R[X; G] = IR[X; G](IR[X; G])^{-1} = R[X; G].$$

So $II^{-1} = R$ and therefore [6, Theorem 4.4] implies that R is a π-domain.

Recall that an integral domain is called a pseudo-principal (resp. pseudo-Dedekind) domain if every divisorial ideal is principal (resp. invertible) ([3]).

COROLLARY 5 ([10, Theorem 8]). *R is a pseudo-princiapl domain and G is of type* $(0, 0, 0, \cdots)$ *if and only if $R[X; G]$ is a pseudo-principal domain.*

Proof. It directly follows Lemma 3.

COROLLARY 6 ([3, Theorem 4.5]). *R is a pseudo-Dedekind domain and G is of type* $(0, 0, 0, \cdots)$ *if and only if $R[X; G]$ is a pseudo-Dedekind doamin.*

Proof. It also directly follows from Lemma 3.

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